Creeping Flow of Power-Law Fluid over Newtonian Fluid Sphere

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A technique which is a combination of Galerkin's method and variational principle was developed and used for the approximate solution of creeping flow of power-law fluid over a Newtonian fluid sphere. The stream functions (both internal and external) and drag coefficient are expressed in terms of three parameters: the flow behavior index of the power law fluid, the external Reynolds number, and a viscosity ratio parameter $X = \frac{\mu_i a^{n-1}}{KV_\infty^{n-1}}$. Comparisons with existing experimental data are also given.

The knowledge of the motion of fluid particles through another fluid medium is of significant importance because of its many applications in numerous fields, and its study perhaps represents one of the more thoroughly investigated topics in literature. The classical treatment on this subject was first made by Hadamard (*10*) and Rybczynski (*18*), who considered the motion to be of creeping flow type. Subsequent works have treated other flow regimes as well as various kinds of complicating factors such as the distortion of the fluid sphere and the effect of surfactants. These include the work of Chao (*4*) and Moore (*15*) for high Reynolds number, the approximate solution of Hamielec and co-workers (*11*), and the solution of Taylor and Acrivos (*20*) which deals with nonspherical fluid particles.

All these references assume that both the external and internal fluids are Newtonian. Until recently no attempts have been made to extend these studies to non-Newtonian media. The consideration of fluid media other than Newtonian, besides its academic interest, would also be of substantial practical importance since some of the fluid systems encountered in liquid-gas contact processes have been found to exhibit non-Newtonian behavior. This includes the fermentation broth used for penicillin production (*6*) and the liquid sludge in activated sludge processes for waste treatment (*1*).

Astarita (*2*) considered the motion of a gas bubble through a power law fluid as well as a viscoelastic fluid. For the case of a power law fluid, the drag coefficient is expressed as

$$C_o = \frac{X_n}{N_{Re_a}} \tag{1}$$

and

$$\left(\frac{X_n}{X_n'}\right)^{\frac{1}{n}} > 1.5 \tag{2}$$

This conclusion was made based on some semiquantitative arguments. Since it is given as an inequality, the result can be used only as a limit. Another implicit assumption in Astarita's work is that the viscosity ratio between the internal fluid and external fluid is negligible. This is justified for the case of gas bubbles moving through a liquid media but would be inadequate for the case where the viscosity of the liquid drop is comparable with that of the outer fluid. Furthermore, Astarita's analysis did not provide any information on the flow pattern both inside and outside the fluid particle which could be required if the related mass transfer problem is to be studied.

Although the study of non-Newtonian flow over fluid spheres has been relatively inactive, considerable interest has been shown in the subject of non-Newtonian flow over rigid spheres. Rathna (*17*) employed a perturbation approximation for the flow of Reiner-Rivlin-Prager fluid over a rigid sphere, assuming both the viscosity and cross viscosity to be constant. Leslie (*13*) considered the creeping flow of rigid spheres through fluids characterized by the five-constant Oldroyd model. Caswell and Schwarz (*5*) studied the case of Rivlin-Ericksen fluids. Both solutions were made possible through the use of perturbation techniques and reduced to the Stoke and Stoke-Oseen solutions, respectively, as the fluid becomes Newtonian.

A popular method which has been frequently used in connection with the creeping flow of non-Newtonian fluid is the application of a variational principle. Tomita (*21*) appears to be the first person who formulated the problem of slow flow of power law fluid over a rigid sphere and obtained an expression for the drag coefficient. More rigorous analyses on the variational theorem for non-Newtonian flow have been given by Johnson (*12*) and Bird (*3*). Subsequent investigators have applied this approach to consider a number of cases (*19, 22* to *24*). In addition to the use of variational principle which leads to the approximate expression of drag coefficient as an upper limit, Wasserman and Slattery (*22*) employed the reciprocal variational theorem and obtained a lower bound of the expression of drag coefficient for the case of power law fluid.

The object of the present investigation is to present an analysis of the creeping flow of a power law fluid over a Newtonian fluid sphere. Unlike the case of rigid sphere, two equations of motion (for the external and internal fluid, respectively) have to be satisfied. This was accomplished through the use of a combination of Galerkin's method and variational integrals. Although only the power law model was considered in the present work, the procedure developed is believed to be applicable to other rheological models with only minor modification.

ANALYSIS

The equations of continuity and motion for an incompressible fluid can be written in the following tensorial form:

$$\frac{\partial \rho}{\partial t} = -(\rho v^j)_{,j} \tag{3}$$

$$\rho\left[\frac{\partial v^i}{\partial t} + v^j v^i_{,j}\right] = -p^{,i} - \tau^{ij}_{,j} + \rho f^i \tag{4}$$

For the present problem, the following assumptions are made:

1. Axisymmetric steady state creeping flow (that is, the inertial terms in the equation of motion can be neglected).
2. The fluid particle is perfectly spherical.
3. No temperature and concentration gradient are present so that all the physical properties are constant.
4. The internal fluid is Newtonian, while the external fluid obeys power law model or

$$\tau_j^i = -\mu\Delta_j^i \text{ for internal fluid} \tag{5a}$$

$$\tau_j^i = -K\left(\frac{1}{2}\Delta_k^m\,\Delta_k^m\right)^{\frac{n-1}{2}} \Delta_j^i \text{ for external fluid} \tag{5b}$$

where Δ_j^i, the rate of deformation tensor, is given as $v^i_{,j} + v^j_{,i}$. With these assumptions, the equation of motion for the internal fluid written in terms of stream function and spherical coordinates (dimensionless) becomes

$$D^4\,\Psi_i = 0 \quad r < 1 \tag{6}$$

where

$$D^4 = \left[\frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right]^2 \tag{7}$$

and

$$(v_r)_i = \frac{1}{r^2\sin\theta}\frac{\partial\Psi_i}{\partial\theta} \tag{8a}$$

$$(v_\theta)_i = \frac{-1}{r\sin}\frac{\partial\Psi_i}{\partial r} \tag{8b}$$

The dimensionless coordinate r is given as R/a. R is the radial distance from the center of the sphere and a is the radius of the sphere. The dimensionless velocity component v is given as V/V_∞, where V is the physical velocity component and V_∞ is the relative velocity between the fluid particle and the external fluid.

If the velocity variational principle is applied to the external fluid, it can be shown (*3, 13, 16*) that the solution of Equation (4) with the assumptions stated before is equivalent to the minimization of the integral

$$\min J_V = \int_{V_o} \Gamma\, d\, V_o \tag{9}$$

The volume V_o is taken as $r > 1$. The integrand Γ is given as

$$\Gamma = \frac{K}{2(n+1)}\left(\frac{1}{2}\Delta_j^i\Delta_j^i\right)_o^{\frac{n+1}{2}} \tag{10}$$

The rate of deformation tensor is to be evaluated based on the velocity profile in the external fluid (or external stream function). The boundary conditions are assumed to be

$$(v_r)_i = (v_r)_o = 0 \text{ at } r = 1 \tag{11a}$$

$$(v_\theta)_i = (v_\theta)_o \text{ at } r = 1 \tag{11b}$$

$$(\tau_{r\theta})_i = (\tau_{r\theta})_o \text{ at } r = 1 \tag{11c}$$

$$\psi_o \to -\tfrac{1}{2}\, r^2 \sin^2\theta \text{ as } r \to \infty \tag{11d}$$

$$(v_\theta)_i \text{ and } (v_r)_i \text{ remains finite at } r = 0 \tag{11e}$$

The problem described above will be solved using a combination of variational principles and Galerkin's method. First, the trial functions for the internal and external stream functions are assumed:

$$\psi_i = (C_1r^2 + C_2r^3 + C_3r^4)(1 - Z^2) + (D_1r^2 + D_2r^3 + D_3r^4)\, Z(1 - Z^2) \tag{12a}$$

$$\psi_o = \left(-\frac{1}{2}r^2 + A_1r^\sigma + \frac{A_2}{r}\right)(1 - Z^2) + \left(\frac{B_1}{r} + \frac{B_2}{r^2}\right) Z(1 - Z^2) \tag{12b}$$

$$Z = \cos\theta \tag{13}$$

The coefficients, A_1, A_2, B_1, B_2 ..., and the exponents σ, remain to be determined. The trial functions are so taken that for the case of $n = 1$, with a proper choice of coefficients, Equations (12*a*) and (12*b*) would become identical with the Hadamard-Bybczynski solution.

Applying the boundary conditions of Equations (11*a*) and (11*b*), one has

$$A_1 + A_2 = \tfrac{1}{2} \tag{14}$$

$$B_1 + B_2 = 0 \tag{15}$$

$$C_1 + C_2 + C_3 = 0 \tag{16}$$

$$D_1 + D_2 + D_3 = 0 \tag{17}$$

$$A_1\sigma - A_2 - 1 = 2C_1 + 3C_2 + 4C_3 \tag{18}$$

$$-B_1 - 2B_2 = 2D_1 + 3D_2 + 4D_3 \tag{19}$$

Applying Galerkin's method for the approximate solution of Equation (6) as well as satisfying the boundary conditions of Equation (11*c*), one has

$$\int_0^1\int_{-1}^1 r(1 - Z^2)\, D^4\psi_i dZ\, dr = 0 \tag{20a}$$

$$\int_0^1\int_{-1}^1 r^2 Z(1 - Z^2)\, D^4\psi_i\, dZ\, dr = 0 \tag{20b}$$

$$\int_{-1}^1 (1 - Z^2)\,[(\tau_{r\theta})_o - (\tau_{r\theta})_i]_{r=1}\, dZ = 0 \tag{21a}$$

$$\int_{-1}^1 Z(1 - Z^2)\,[(\tau_{r\theta})_o -, (\tau_{r\theta})_i]_{r=1}\, dZ = 0 \tag{21b}$$

Combining Equations (12*a*), (21*a*), and (21*b*) yields

$$C_2 = 0 \tag{22a}$$

$$D_3 = 3D_1 \tag{22b}$$

The evaluations of the integrals of Equations (21*a*) and (21*b*) present certain difficulties because of the nonlinear term

$$\left(\frac{1}{2}\Delta_j^i\Delta_j^i\right)_o^{\frac{n-1}{2}}\Bigg|_{r=1}$$

associated with the shear stress term $(\tau_{r\theta})_o$. To circumvent this difficulty, this term is written as a Fourier expansion:

$$(\phi_o)_{r=1} = \left[\left(\frac{a}{V_\infty}\right)^2\left(\frac{1}{2}\Delta_j^i\Delta_j^i\right)\right]^{\frac{n-1}{2}}\Bigg|_{r=1}$$

$$= \frac{\alpha_o}{2} + \sum_{j=1}^{\infty} \alpha_j \cos i\theta \quad (23)$$

The expansion coefficients α_o, α_1, ... can be evaluated in the routine manner such as

$$\alpha_o = \frac{2}{\pi}\int_o^\pi \phi_o\Big|_{r=1} d\theta \quad (24a)$$

$$\alpha_i = \frac{2}{\pi}\int_o^\pi \phi_o\Big|_{r=1} \cos i\theta \; d\theta \quad (24b)$$

Equation (23) can be rearranged and written in terms of a polynomial in $\cos\theta$ (or Z). If only a finite number of terms is taken (say five), this becomes

$$\phi_o|_{r=1} = \overline{\alpha}_o + \overline{\alpha}_1 Z + \overline{\alpha}_2 Z^2 + \overline{\alpha}_3 Z^3 + \overline{\alpha}_4 Z^4 \quad (25)$$

where

$$\overline{\alpha}_o = \frac{\alpha_o}{2} - \alpha_2 + \alpha_4 \quad (26a)$$

$$\overline{\alpha}_1 - \alpha_1 - 3\alpha_3 \quad (26b)$$

$$\overline{\alpha}_2 = 2(\alpha_2 - 4\alpha_4) \quad (26c)$$

$$\overline{\alpha}_3 = 4\alpha_3 \quad (26d)$$

$$\overline{\alpha}_4 = 8\alpha_4 \quad (26e)$$

Based on the expression of $\phi_o|_{r=1}$ given by Equation (21) and the assumed trial function of ψ_i and ψ_o [Equations (12*a*) and (12*b*)], Equations (21*a*) and (21*b*) become

$$\frac{s}{2}[1 + (\sigma - 3)\sigma A_1 + 4A_2] + t(3B_2) = 3C_3 + C_2 \quad (27a)$$

$$\frac{l}{2}[1 + (\sigma - 3)\sigma A_1 + 4A_2] + m(3B_2) = 2D_1 + 3D_2 + 5D_3 \quad (27b)$$

where

$$s = \frac{1}{48X}(48\overline{\alpha}_o + 8\overline{\alpha}_2 + 3\overline{\alpha}_4) \quad (28a)$$

$$t = \frac{1}{48X}(8\overline{\alpha}_1 + 3\overline{\alpha}_3) \quad (28b)$$

$$l = \frac{1}{8X}(8\overline{\alpha}_1 + 3\overline{\alpha}_3) \quad (28c)$$

$$m = \frac{1}{16X}(16\overline{\alpha}_o + 6\overline{\alpha}_2 + 3\overline{\alpha}_4) \quad (28d)$$

$$X = \frac{\mu_i a^{n-1}}{KV_\infty^{n-1}} \quad (29)$$

By some algebraic manipulation, ten of the eleven undetermined parameters (A_1, A_2, B_1, B_2, C_1, C_2, C_3, D_1, D_2, D_3, and σ) can be eliminated for the ten equations [Equations (14) to (19), (21*a*) and (21*b*), and (27*a*) and (27*b*)]. These are given as

$$A_1 = \frac{3[15lt - (6q + 5s)(3m + 2q)]}{2(\sigma + 1)[15lt + (6q + 5s)\{q(\sigma - 4) - 3m\}]} \quad (30a)$$

$$A_2 = \frac{1}{2} - A_1 \quad (30b)$$

$$B_1 = -B_2 = \frac{-6l}{6q + 5s}\left[\frac{3}{4} - \frac{1}{2}(\sigma + 1)A_1\right] \quad (30c)$$

$$C_1 = -C_3 = \frac{3}{4} - \frac{1}{2}(\sigma + 1)A_1 \quad (30d)$$

$$C_2 = 0 \quad (30e)$$

$$D_1 = \frac{-3l}{6q + 5s}\left[\frac{3}{4} - \frac{1}{2}(\sigma + 1)A_1\right] \quad (30f)$$

$$D_2 = \frac{12l}{6q + 5s}\left[\frac{3}{4} - \frac{1}{2}(\sigma + 1)A_1\right] \quad (30g)$$

$$D_3 = \frac{-9l}{6q + 5s}\left[\frac{3}{4} - \frac{1}{2}(\sigma + 1)A_1\right] \quad (30h)$$

where

$$q = ms - lt \quad (30i)$$

In other words, all ten coefficients are expressed in terms of the exponent σ. Actually this is not exactly correct since the parameters l, s, m, ... are also dependent upon the coefficients A_1, A_2, ... as well as σ [see Equations (28*a*) to (28*d*), (26*a*) to (26*e*) and (24*a*) and (24*b*)]. However, as shown in a later section, the numerical computation started with the Newtonian cases ($n = 1$) and proceeded with decreasing n (say $n = 0.975$, 0.95, etc.). For the first iteration, the nonlinear term $\phi_o|_{r=1}$ was estimated for $n = n_o$ using the values of stream functions for $n = n_o + \Delta n$. Consequently, all the quantities, l, s, m, t, and q, are regarded as constants in the above equations.

With the coefficients A_1, A_2, ... expressed in terms of the exponent σ, the problem reduces to the determination of σ so that the integral of Equation (9) is a minimum. First, Equation (9), is rewritten as

$$J_V = \int_{V_o} \Gamma \, dV_o$$

$$= + \frac{K}{2(n+1)}\int_{V_o}\left[\frac{1}{2}(\Delta_j^i\Delta_j^i)\right]^{\frac{n+1}{2}} \cdot dV_o \quad (31)$$

Since

$$\frac{1}{2}(\Delta_j^i\Delta_j^i)_o = \frac{V_\infty^2}{a^2}\phi_o^{\frac{2}{n-1}} \quad (32)$$

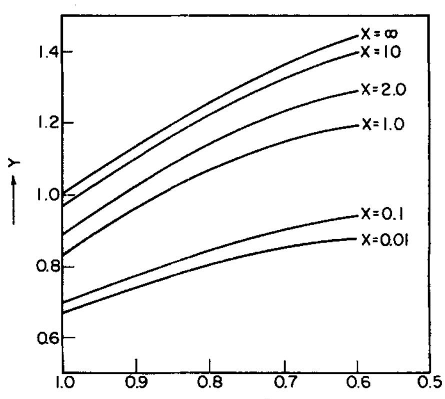

Fig. 1. Calculated values of the correction factor Y vs. flow behavior index *n*.

TABLE 2. VALUES OF E

n	$X = 0.1$ E	Δn
0.975	3.025×10^{-3}	0.025
0.950	6.204×10^{-3}	0.025
0.900	1.346×10^{-2}	0.050
0.850	2.167×10^{-2}	0.050
0.800	3.178×10^{-2}	0.050
0.750	4.351×10^{-2}	0.050
0.700	5.107×10^{-2}	0.050
0.650	6.586×10^{-2}	0.050
0.600	7.337×10^{-2}	0.050

and

$$\phi_o = \Bigg[\frac{2}{r^4 \sin^2\theta}\left(\frac{2}{r}\frac{\partial \psi_o}{\partial \theta} - \frac{\partial^2 \psi_o}{\partial r \partial \theta}\right)^2 + \left(\frac{\partial^2 \psi_o}{\partial r \partial \theta} - \cot\theta \frac{\partial \psi_o}{\partial r} - \frac{1}{r}\frac{\partial \psi_o}{\partial \theta}\right)^2 + \left(\frac{1}{r}\frac{\partial \psi_o}{\partial \theta} - \cot\theta\frac{\partial \psi_o}{\partial r}\right)^2 + \frac{1}{2}\left(r\frac{\partial^2 \psi_o}{\partial r^2} - 2\frac{\partial \psi_o}{\partial r} + \frac{1}{r}\cot\theta\frac{\partial \psi_o}{\partial \theta} - \frac{1}{r}\frac{\partial^2 \psi_o}{\partial \theta^2}\right)^2\Bigg]^{\frac{n-1}{2}} \quad (33)$$

$$J_V = \frac{K}{2(n+1)}\left(\frac{V_\infty^2}{a^2}\right)^{\frac{n+1}{2}} \int_{V_o}^{\pi} \phi_o^{\frac{n+1}{n-1}} \cdot dV_o$$

$$= \frac{K}{2(n+1)}\left(\frac{V_\infty^2}{a^2}\right)^{\frac{n+1}{2}} (a^3)(2\pi)\int_1^\infty \int_o^\pi \phi_o^{\frac{n+1}{n-1}} r^2 \cos\theta \cdot dr\, d\theta$$

$$= \frac{\pi K}{(n+1)} \frac{V_\infty^{n+1}}{a^{n+1}} a^3 \int_{-1}^1 \int_o^1 \phi_o^{\frac{n+1}{n-1}} x^{-4}\, dx\, dZ \quad (34)$$

where

$$x = \frac{1}{r}, \qquad Z = \cos\theta \quad (35)$$

The integrand $x^{-4}\phi_o^{\frac{n+1}{n-1}}$ can be evaluated by substituting Equation (12*b*) into (33). This is given as

$$x^{-4}\phi_o^{\frac{n+1}{n-1}} = \left[\sum_{i=1}^{4} (J_i)^2\right]^{\frac{n+1}{2}} \quad (36)$$

where

$$A_1' = \frac{dA_1}{d\sigma} = -A_2' = \frac{[15lt - (6q+5s)(3m+2q)]\,[15lt + (6q+5s)\{q(\sigma-4) - 3m\} + (\sigma+1)(6q+5s)q]}{2(\sigma+1)^2\,[15lt + (6q+5s)\{q(\sigma-4) - 3m\}]^2} \quad (43a)$$

$$J_1 = x^{\frac{2(n-1)}{n+1}}\,[2\,ZW + B_1(3Z^2-1)(3x^2 - 4x^3)] \quad (37a)$$

$$J_2 = x^{\frac{2(n-1)}{n+1}}\,[-ZW + B_1(2-5Z^2)\,x^2 - (3-7Z^2)x^3] \quad (37b)$$

$$J_3 = x^{\frac{2(n-1)}{n+1}}\,[ZW + B_1(4Z^2-1)x^2 - (5Z^2-1)x^3] \quad (37c)$$

$$J_4 = \frac{1}{\sqrt{2}} x^{\frac{2(n-1)}{n+1}} (1-Z^2)^{1/2}\,[(\sigma^2 - 3\sigma + 2)A_1 x^{1-\sigma} + 6A_2x^2 + ZB_1(10x^2 - 16x^3)] \quad (37d)$$

$$W = (2-\sigma)\,A_1x^{1-\sigma} + 3A_2x^2 \quad (38)$$

$$J_V = \frac{\pi K}{(n+1)} V_\infty^{n+1} a^{2-n} \int_{-1}^1 \int_o^\infty \left[\sum_{i=1}^4 (J_i)^2\right]^{\frac{n+1}{2}} \cdot dx\, dZ \quad (39)$$

The problem now is to minimize the integral J_V with respect to the undetermined exponent σ, since all the coefficients $A_1, A_2, \ldots$ are expressed in terms of σ [see Equations (30*a*) to (30*h*)]. The necessary condition for the minimization of J_V is simply given as

$$\frac{\partial J_V}{\partial \sigma} = 0$$

or

$$\int_{-1}^1 \int_o^1 \left(\frac{n+1}{2}\right)\left[\sum_{i=1}^4 (J_i)^2\right]^{\frac{n-1}{2}} \left[2\sum_{i=1}^4 J_i \frac{\partial J_i}{\partial \sigma}\right] dx\, dZ = 0 \quad (40)$$

and the derivatives $\partial J_i/\partial \sigma$ are given as

$$\frac{\partial J_1}{\partial \sigma} = x^{\frac{2(n-1)}{n+1}}\,[2Z\,\xi + (3Z^2-1)(3B_1'x^2 + 4B_2'x^3)] \quad (41a)$$

$$\frac{\partial J_2}{\partial \sigma} = x^{\frac{2(n-1)}{n+1}}\,[-Z\,\xi + (2-5Z^2)\,B_1'x^2 + (3-7Z^2)\,B_2'x^3] \quad (41b)$$

$$\frac{\partial J_3}{\partial \sigma} = x^{\frac{2(n-1)}{n+1}}\,[Z\,\xi + (4Z^2-1)\,B_1'x^2 + (5Z^2-1)\,B_2'x^3] \quad (41c)$$

$$\frac{\partial J_4}{\partial \sigma} = \frac{1}{\sqrt{2}} x^{\frac{2(n-1)}{n+1}} \sin\theta\,[\{2\sigma - 3\}\,A_1 + (\sigma^2 - 3\sigma + 2)A_1'\}x^{1-\sigma} - (\sigma^2 - 3\sigma + 2)x^{1-\sigma}\ln x + 6A_2'\,x^2 + Z(10B_1'\,x^2 + 16B_2'\,x^3)] \quad (41d)$$

$$\xi = [-A_1 + (2-\sigma)\,A_1']\,x^{1-\sigma} - (2-\sigma)A_1x^{1-\sigma}\ln x + 3A_2'\,x^2 \quad (42)$$

$$B_1' = -B_2' = \frac{6l}{6q+5s}\left[\frac{1}{2}A_1 + \frac{1}{2}(\sigma+1)A_1'\right] \quad (43b)$$

Numerical values of σ can be evaluated for specified values of n and X. Based on the values of σ, one can eval-

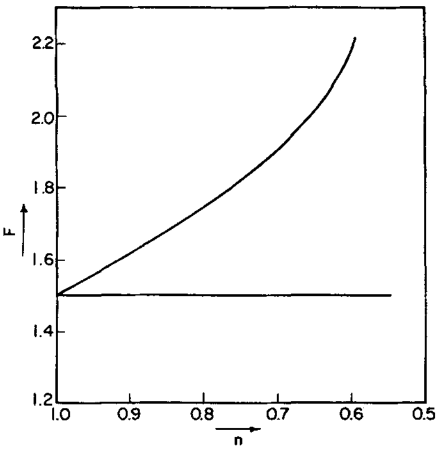

Fig. 2. *F* vs. *n*.

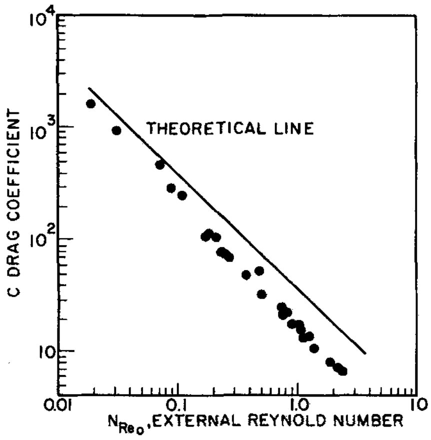

Fig. 3. Comparison of experimental data of Fararoui and Kintner with this work (CMC solution, $n = 0.88$).

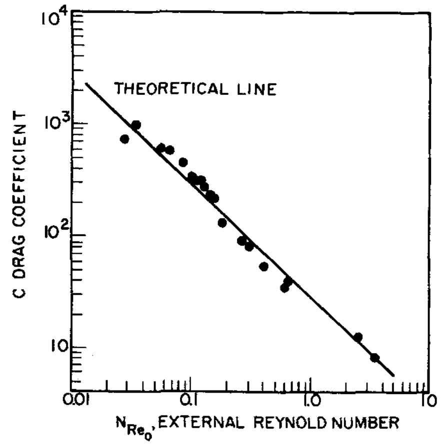

Fig. 4. Comparison of experimental data of Fararoui and Kintner with this work (Lytrol solution, $n = 0.56$).

uate the coefficients A_1, A_2, ..., which, in turn, give the stream functions ψ_i and ψ_o. Also, with the stream functions, the total drag coefficient can be computed. This is given as

$$C = \frac{F_k}{\left(\frac{1}{2}\right) V_\infty^2 \rho_o (\pi a^2)} = \frac{\int_{V_i} \tau_j^i V^i_{,j} dV_i + \int_{V_o} \tau_j^i V^i_{,j} dV_o}{\left(\frac{1}{2}\right) V_\infty^2 \rho_o (\pi (a^2))} \tag{44}$$

In an alternate form, one can express C as

$$C = \frac{24Y}{2^n N_{Re_o}} \tag{45}$$

where

$$N_{Re_o} = \frac{\rho_o V_\infty^{2-n} a^n}{K} \tag{46}$$

One can regard Y as a correction factor and is given as

$$Y = \frac{2^{n-2} a^{n-2}}{3K\pi V_\infty^{n+1}} \left[\int_{V_i} \mu_i \left(\frac{1}{2} \Delta_j^i \Delta_j^i \right)_i dV_i + \int_{V_o} K \left(\frac{1}{2} \Delta_j^i \Delta_j^i \right)^{\frac{n+1}{2}} dV_o \right]$$
$$= \frac{2^{n+1}}{3} \int_{-1}^{1} \int_{o}^{1} \left(X\, x^2 \phi_i^{\frac{2}{n-1}} + \phi_o^{\frac{n+1}{n-1}} x^{-4} \right) dx\, dZ \tag{47}$$

Where ϕ_o is given by Equation (33). ϕ_i can be written in similar manner based on ψ_i. The correction factor is given as a function of n and X.

RESULTS AND DISCUSSION

The numerical computation of this investigation centers on the solution of Equation (40) which yields the value of σ for a specified value of X and n and is carried out using an iterative procedure in the following manner: The numerical calculation started with $n = 0.975$. At first, the parameters l, s, m, t, ... [Equations (28*a*) to (28*d*)] were estimated based on the solution with $n = 1$ (Hadamard-Rybczynski solution) for a given value of X. With the values of l, s, m, t, and q known, Equation (40) is solved by the method of regula falsi using Simpson's formula for the evaluation of the double integral. Based on the value of σ obtained from Equation (40), together with the estimated values of l, s, m, t, and q, the coefficients A_1, A_2, ... can be evaluated accordingly, which leads to a second estimation of the values of l, s, m, t, ... The same procedure is repeated to evaluate the coefficients A_1, A_2, ... This iteration is carried out until the values of the coefficients A_1, A_2, ... obtained from successive computations become essentially the same. The same procedure was used for the case of $n = 0.95$ with l, s, m, t, and q first estimated based on the solution with $n = 0.975$. This process was repeated with decreasing values of flow behavior index. The range of parameters is given as

$$0.001 < X < 10.0$$
$$0.5 < n < 1.0$$

It was also found that almost identical results were found for $X = 0.001$ and $X \leqq 0.0001$. Consequently, the values for $X = 0.001$ are taken as a lower limit. Numerical results expressed in the form Y [Equation (47)] and σ vs. n with specified values for X are given in Figure 1. Numerical values of the coefficient A_1, A_2, ... of the stream functions for selective cases are given in Table 1.*

A possible source of error in the present analysis is that only finite numbers were considered in the Fourier series expression for the term $(\phi_o)_{r=1}$, Equation (22). Consequently the continuity condition on the shear stress at the fluid-fluid interface cannot be maintained exactly. An error-indicating quantity can be defined as

$$E^2 = \int_S [(\tau_{r\theta})_o - (\tau_{r\theta})_i]^2 \, dS \tag{48}$$

where the internal and external shear stresses can be evaluated according to Equations (5*a*) and (5*b*), respectively. The surface integral is carried out through the entire surface of fluid sphere.

Numerical values of E for a variety of cases have been calculated. Table 2 presents values of E at various values of n for $X = 0.1$. These values on the whole are of low magnitude but increase rapidly with the decrease of n (more pronounced non-Newtonian behavior). These calculations seem to indicate that procedures used in this work for the approximate linearization of non-Newtonian term is a reasonable one as long as the non-Newtonian behavior is not too pronounced.

As stated previously, Astarita (2) obtained a semi-quantitative relationship for the drag coefficient of gas spheres in creeping motion through a power law fluid. In

* Tabular material has been deposited as document 9794 with the American Documentation Institute, Photoduplication Service, Library of Congress, Washington, D.C., 20540 and may be obtained for $1.25 for photoprints or 35-mm. microfilm.

terms of the correction factor Y, his relationship can be expressed as

$$F = \left(\frac{Y_{\text{rigid sphere}}}{Y_{\text{fluid sphere}}} \right)^{1/n} > 1.5 \qquad (49)$$

$$X = 0$$

Values of F for various values of n were calculated and are shown in Figure 2. The inequality condition is observed for all cases.

Drag coefficients of liquid drops flowing through polymer solutions have been determined experimentally by Mhatre and Kintner (*14*) and Fararoui and Kintner (*7*). The experiments were conducted by measuring the terminal velocity of liquid drops of nitrobenzene and tetrachloroethane mixture falling through aqueous solution of carboxymethylcellulose (CMC) and Lytron 890. Comparisons between the experimental data and present analysis in the form of total drag coefficient vs. Reynolds number are presented in Figures 3 and 4. For their comparisons, the parameter X is estimated to be in the order of $10^{-4} \sim 10^{-3}$ and therefore was assumed to be zero.

The agreement between the experimental and theoretical results is good for the case of CMC solution (n = 0.88). For the case of Lytron 890, the experimental points were consistently below the theoretical line and the deviation appears to be more significant at high values of N_{Re}. The precise reasons which cause this are not known, although several arguments can be advanced. First, the condition of continuity of stress at interface is only satisfied approximately. Also, the drag coefficient obtained from the variational principle such as used in this work is known to be only an upper bound. This seems to be in qualitative agreement with the comparison as shown in Figure 4. Attempts were made to obtain a lower bound expression of drag coefficient based on reciprocal principle. This however was found unsuccessful because of the unprescribed velocity at the fluid-fluid interface.

In applying the results of present work to practical physical problems, it should be noted that the particles are not always spherical as assumed in this analysis except for extreme small bubbles or drops. Haberman and Morton (*9*) observed experimentally that an increase in the size of the fluid particle is accompanied by a change of the shape from spherical to ellipsoidal to a spherical cap. A similar phenomenon has also been observed (*7, 14*) for non-Newtonian cases. To the author's knowledge, no studies have been carried out in the past to consider the deformation of fluid particles in a non-Newtonian medium because of the extreme mathematical complexities involved. On the other hand, if the effect of particle deformation on the drag is assumed to be the same for both the Newtonian and non-Newtonian cases, the results of Taylor and Acrivos (*20*) can be readily adapted. The correction factor for the drag coefficient is given as

$$(C - C_{\text{spherical}}) \frac{N_{Re}}{2\pi} = \lambda \frac{3X^2 - X + 8}{5(X+1)^2} N_{We} \qquad (50)$$

where N_{We}, the Weber number, is given as

$$N_{We} = \rho_a V_\infty^2 / \gamma \qquad (51)$$

and the parameter λ is given as

$$\lambda = \frac{1}{4(X+1)^3} \left[\left(\frac{81}{80} X^3 + \frac{57}{20} X + \frac{103}{40} K + \frac{3}{4} \right) - \frac{\rho_i - \rho_o}{12\rho_i} (X+1) \right] \qquad (52)$$

In using this correction factor, it should be kept in mind that this result was obtained on the basis of low Reynolds number as well as small Weber's number.

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NOTATION

A_1, A_2 = coefficients of external stream function
A_i' = $dA_i/d\sigma$
a = radius of fluid sphere
B_1, B_2 = coefficients of external stream function
B_i' = $dB_i/d\sigma$
C_o defined by Equation (1)
$C_1\, C_2\, C_3$ = coefficients of internal stream function
C defined by Equation (44)
D_1, D_2, D_3 = coefficients of internal stream function
E defined by Equation (48)
F defined by Equation (49)
F_k = force acting on fluid sphere
f^i = body force vector
J_i defined by Equation (37)
J_v defined by Equation (9)
K = consisting index of the power law fluid
l, m, s, t defined by Equation (28)
n = flow behavior index
P = pressure
q defined by Equation (30*i*)
R = radial distance from the center of the sphere
r = dimensionless coordinate given as R/a
N_{Re_a} = $(2a)^n \rho V_\infty^{2-n}/K$
N_{Re_o} = $a^n \rho_o V_\infty^{2-n}/K$
N_{We} = $\rho_a V_\infty^2/\gamma$
S = boundary surface
V = physical velocity vector
V_∞ = relative velocity between fluid sphere and field fluid
V_i = region where $r < 1$
V_o = region where $r > 1$
v = dimensionless velocity vector in Equations (1) and (2)
v_r, v_θ = dimensionless velocity components
W defined by Equation (38)
X = $\mu_i a^{n-1}/KV_\infty)^{n-1}$
x = r^{-1}
X_n defined Equation (II-5) in reference *1*
X_n' defined Equation (II-10) in reference *1*
Y defined by Equation (47)
Z = $\cos\theta$

Greek Letters

α_i defined by Equation (24)
$\bar{\alpha}_i$ defined by Equation (26)
Γ defined by Equation (10)
Δ_j^i = rate of deformation tensor
ξ defined by Equation (42)
σ = coefficient of external stream function
γ = interfacial tension
μ = viscosity
θ = angle
ρ = density
τ_j^i = stress tensor
ϕ_o defined by Equation (33)
ψ = stream function

Subscripts

i = internal fluid
o = external fluid

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Direct Contact Heat Transfer Between Immiscible Liquids in Turbulent Pipe Flow

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This paper reports on an experimental study of the direct contact heat transfer between oil and water in turbulent pipe flow under nonboiling conditions. Data were taken by a new technique, namely, monitoring on a very fast response recorder the output of a small thermocouple placed in the two-phase flow. The variables studied were the liquid velocity, the pipe diameter, the water volume fraction, and, to a lesser degree, the interfacial tension and the oil viscosity. A successful semiempirical method of correlating the data is also presented.

A new saline water conversion process, the cardinal feature of which is heating brine by dispersing it in a hot immiscible liquid in highly turbulent pipe flow, was proposed by Wilke (*13*) in 1958. Since then, various aspects of this process have been under investigation at the Sea Water Conversion Laboratory of the University of California at Berkeley, and a survey of these investigations has been given in a recent paper by Wilke et al. (*14*). That paper suggested possible advantages for a direct contact heat transfer process for seawater conversion over conventional distillation processes using metallic heat transfer surfaces, and went on to outline how the basic idea might be employed in developing a multistage seawater conversion plant. The potential advantages include the abatement of scaling and corrosion problems, the use of simple and inexpensive equipment, and the adaptability of the process to very large throughputs.

The present paper reports on an experimental study of the *direct contact* heat transfer between oil and water in turbulent pipe flow under nonboiling conditions. Data were taken by a new technique, namely, monitoring on a very fast response recorder the output of a small thermocouple placed in the two-phase flow. Evidence is presented that the (consistent) method adopted for interpreting the thermocouple traces gives reliable liquid temperatures. The variables studied were the liquid velocity, the pipe diameter, the water volume fraction, and, to a lesser degree, the interfacial tension and the oil viscosity. A successful semiempirical method of correlating the data is also presented. Some measurements of the heat transfer during boiling, the pressure drop in two-phase liquid-liquid flow, and the identity of the continuous phase were also made, but these are only touched upon here. The interested reader may consult the thesis by Porter (*7*) for the details.

The most closely related published work is that of Grover and Knudsen (*1*), who measured the rates of heat transfer between a petroleum solvent and water flowing cocurrently in a 1.50-in. diameter horizontal pipe at total flow rates ranging from 2,500 to 15,000 lb./hr. These flow rates are sufficiently low that stratification occurred within the pipe downstream from the mixing section. Heat transfer coefficients were computed from the liquid temperatures measured in the stratified region. Grover and Knudsen found that for their range of variables the volumetric heat transfer coefficient was independent of the dispersed phase volume fraction and the inlet temperatures, but varied roughly proportional to the 1.6 power of the total linear velocity. In the present study the liquid velocities were sufficiently high to preclude stratification. It should not be surprising then if the dependence of the volumetric heat transfer coefficient on the operating variables is different in this study from that found by Grover and Knudsen (1) or by Wilke et al. (*14*) with stratified